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PLANAR STRONGLY WELL-COVERED GRAPHS

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Introduction.

Plummer [11] introduced the concept of a well-covered graph in 1970. A graph is well-covered if every maximal independent set (with respect to set inclusion) in the graph is also a maximum independent set. Various subclasses of well-covered graphs have been studied (see, for example, [1] - [7], [10], and [12] - [14]). We consider the subclass which we call strongly well-covered graphs. A strongly well-covered graph G is a well-covered graph with the additional property that G-e is also well-covered for every edge e in G. By making use of (i) structural characteristics of strongly well-covered graphs and (ii) the theory of Euler contributions (for planar graphs), we show that there are only four planar strongly well-covered graphs.

Preliminaries.

From the definition, strongly well-covered graphs remain well-covered upon deletion of any edge. Well-covered graphs which remain well-covered upon deletion of any vertex (called 1-well-covered) have previously been studied by several authors (see [10], [13] and [14]). It is interesting to note that a strongly well-covered graph fails to remain well-covered if any vertex is deleted. The following theorem is proved in [10].

Theorem 1. If G ($G \neq K_1$ or K_2) is strongly well-covered, then for all vertices v in G the graph G-v is not well-covered .

Two structural characteristics which we need are stated in the following two theorems. The proof of 3-connectedness proceeds by induction on the independence number. See [9] or [10] for proofs.

Theorem 2. If G is strongly well-covered, $G \notin \{K_1,K_2,C_4\}$, then $\delta \geq 4$.

Theorem 3. Suppose G is strongly well-covered, $G \notin \{K_1,K_2,C_4\}$. Then G is 3-connected.

Next we state a lemma which we will frequently use later. See [9] or [10] for the proof.

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Lemma 4. Suppose G is well-covered. Also suppose that S is an independent set and x is a point in G such that (i) $x \notin S$ and $x \sim v$ for exactly one v in S, and (ii) S dominates N[x], the closed neighborhood of x. Then G-e is not well-covered, where e = vx.

Let G_v be the subgraph of G obtained from G by deleting a vertex v and all its neighbors. The next lemma states that if the vertex a is isolated in the graph G_v, then the vertices a and v must have the same set of neighbors in G. The proof is by induction on the independence number; see [9] or [10].

Lemma 5. Suppose G is connected and strongly well-covered and v is a point in G such that G_v has an isolated point a. Then $N_G(a) = N_G(v)$.

Planar Strongly Well-covered Graphs.

For the remainder of this paper, we restrict ourselves to planar strongly well-covered graphs. For graphs drawn in the plane, we say two faces are adjacent if they share an edge. If a face F contains vertex v, we say F is incident to v. The size of a face is the number of vertices it contains. We refer to the order and sizes of the faces incident to a vertex v as the face configuration at v.

In the next two lemmas, we consider points of degree four and five, respectively, in planar strongly well-covered graphs.

Lemma 6. Suppose G is strongly well-covered planar and 3-connected. If G has a point of degree four which is on a triangular face, then G is the octahedron graph (see Figure 1).

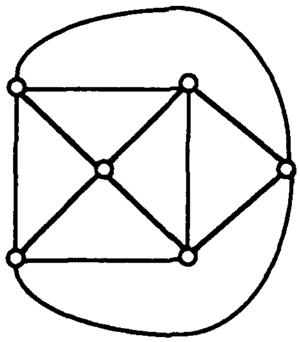

Figure 1

Proof. Suppose v is a point of degree four in G and v is on a triangular face. Let $N(v) = \{u_1,u_2,u_3,u_4\}$. Note that $\delta \geq 4$ by Theorem 2.

Case 1. Suppose the face configuration at v is (3,3,3,3). Let u_1u_2v, u_2u_3v, u_3u_4v and u_4u_1v be the faces.

If $u_1 \sim u_3$, then $\{u_1\}$ dominates N[v]. By Lemma 4, the graph $G\text{-}vu_1$ is not well-covered This contradicts the assumption that G is strongly well-covered. So u_1 is not adjacent to u_3.

Thus, there exists $w \sim u_1$ such that $w \notin \{u_2,u_3,u_4,v\}$.

If w is not adjacent to u_3, then $\{w,u_3\}$ dominates N[v], w is not adjacent to v and $u_3 \sim v$. This leads to a contradiction via Lemma 4. So $w \sim u_3$.

Let $z \sim u_2$ such that $z \notin \{u_1,u_3,u_4,v\}$. If $z \neq w$, then $\{z,u_4\}$ is independent and dominates N[v], z is not adjacent to v and $u_4 \sim v$. By Lemma 4, this is a contradiction. Thus $z = w$; that is, $w \sim u_2$ and $\deg(u_2) = 4$. Similarly, $w \sim u_4$ and $\deg(u_4) = 4$. It then follows that $\deg(u_1) = 4 = \deg(u_3)$. Hence, G is the graph given in Figure 1.

Case 2. Suppose the face configuration at v is (3,3,3,n), $n \geq 4$. Assume the triangular faces are u_2u_3v, u_3u_4v and u_4u_1v. Since G is 3-connected, then u_1 is not adjacent to u_2.

If $u_1 \sim u_3$, then $\{u_3\}$ dominates N[v], a contradiction by Lemma 4. So u_1 is not adjacent to u_3.

Since $\deg(u_1) \geq 4$, there exist points a and b adjacent to u_1 such that $\{a,b\} \cap \{v,u_2,u_3,u_4\} = \varnothing$.

If a is not adjacent to u_3, then $\{a,u_3\}$ is independent and dominates N[v], a is not adjacent to v and $u_3 \sim v$. By Lemma 4, we have a contradiction. So $a \sim u_3$ and, by symmetry, $b \sim u_3$.

Since $\deg(u_2) \geq 4$, there exists $z \sim u_2$ such that $z \notin \{v,u_3,b\}$. Since G is planar, $\{z,u_4\}$ is independent. Then $\{z,u_4\}$ dominates N[v], $u_4 \sim v$ and z is not adjacent to v, a contradiction by Lemma 4.

Thus, the face configuration (3,3,3,n), $n \geq 4$, cannot occur.

Case 3. Suppose the cyclic face configuration is (3,3,m,n), m, n ≥ 4. Assume the triangular faces are u_2u_3v and u_3u_4v. Since G is 3-connected, then u_1 is not adjacent to u_2 and u_1 is not adjacent to u_4.

If $u_1 \sim u_3$, then $\{u_3\}$ dominates N[v], a contradiction by Lemma 4. So u_1 is not adjacent to u_3.

Thus, let $N(u_1) \supseteq \{v,a,b,c\}$, where $\{a,b,c\} \cap \{u_2,u_3,u_4\} = \varnothing$.

If a is not adjacent to u_3, then $\{a,u_3\}$ is independent and dominates N[v], a is not adjacent to v and $u_3 \sim v$. We obtain a contradiction via Lemma 4. So $a \sim u_3$; by symmetry, $b \sim u_3$, $c \sim u_3$.

Without loss of generality, we can assume that b is on the "outside" of cycle $au_1vu_4u_3$ and on the "outside" of cycle $u_1cu_3u_2v$ (see Figure 2). Since $\deg(u_2) \geq 4$, there exists $t \sim u_2$ such that $t \notin \{v,c,u_3\}$. But then $\{b,t,u_4\}$ is independent and dominates $N[v]$, $u_4 \sim v$ and neither b nor t is adjacent to v. So by Lemma 4, we obtain a contradiction.

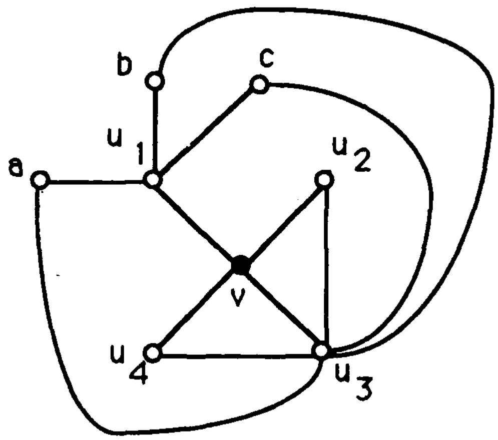

Figure 2

Hence, the cyclic face configuration (3,3,m,n), m, n $\geq$ 4, cannot occur.

Case 4. Suppose the cyclic face configuration at v is (3,m,3,n), m, n $\geq$ 4, with triangular faces u_1u_2v and u_3u_4v. Since G is 3-connected, then u_1 is not adjacent to u_4 and u_2 is not adjacent to u_3.

Case 4.1. Suppose $u_1 \sim u_3$. If there exists $x \sim u_4$ $(x \notin \{v,u_3\})$ such that x is not adjacent to u_1, then $\{x,u_1\}$ is independent and dominates $N[v]$, x is not adjacent to v and $u_1 \sim v$. By Lemma 4, we have a contradiction. Thus, $N(u_1) \supseteq N(u_4)$. Similarly, $N(u_3) \supseteq N(u_2)$. By Lemma 5, it follows that $N(u_1) = N(u_4)$ and $N(u_3) = N(u_2)$. Since $u_1 \sim u_3$ and G is planar, then u_2 is not adjacent to u_4. But $u_3 \sim u_4$, and so $N(u_3) \neq N(u_2)$, a contradiction.

Hence, u_1 is not adjacent to u_3. By symmetry, u_2 is not adjacent to u_4. Thus, there exist points a and b such that a and b are neighbors of u_1 and $\{a,b\} \cap \{v,u_2,u_3,u_4\} = \varnothing$.

Case 4.2. Suppose $a \sim u_2$. If a is not adjacent to u_3, then $\{a,u_3\}$ is independent and dominates $N[v]$, a contradiction by Lemma 4. So $a \sim u_3$ and, similarly, $a \sim u_4$. By Lemma 5, it follows that $N(a) = N(v)$, and so $\deg(a) = 4$.

Since $\delta \geq 4$ and G is planar, then $\{u_1,u_4\}$ is a cutset for G. Since G is 3-connected, we have a contradiction.

Hence, <u>a is not adjacent to u_2</u>. More generally, if $x \sim u_1$, $x \neq v$, then x is not adjacent to u_2. By symmetry, if $y \sim u_2$, $y \neq v$, then y is not adjacent to u_1. Since $\deg(u_i) \geq 4$ for all i, there exist neighbors c and d of u_2 such that $\{c,d\} \cap \{v,u_1\} = \emptyset$, and by the preceding sentence we note that $\{a,b\} \cap \{c,d\} = \emptyset$.

Since G is planar, then x is not adjacent to y for some $x \in \{a,b\}$, $y \in \{c,d\}$. Without loss of generality, suppose b is not adjacent to c.

Case 4.3. Suppose $c \sim u_3$.

Case 4.3.1. If $c \sim u_4$, then $\{c,u_1\}$ is independent and dominates N[v], a contradiction by Lemma 4. So c is not adjacent to u_4.

Case 4.3.2. If b is not adjacent to u_4, then $\{b,c,u_4\}$ is independent and dominates N[v], a contradiction. So $b \sim u_4$.

Case 4.3.3. If $b \sim u_3$, then $\{b,u_2\}$ dominates N[v], a contradiction. Thus, b is not adjacent to u_3.

Case 4.3.4. Suppose $u_4 \sim x$ for all $x \in N(u_1)-u_2$. Then $\{u_2,u_4\}$ is independent and dominates $N[u_1]$, a contradiction by Lemma 4. So there exists $x \sim u_1$, $x \neq u_2$, such that x is not adjacent to u_4.

If x is not adjacent to c, then $\{c,x,u_4\}$ is independent and dominates N[v], a contradiction. So $x \sim c$.

Now by *symmetry of the points* u_1 *and* u_2, *there exists* $y \sim u_2$, $y \neq u_1$, such that y is not adjacent to u_3. Since $x \sim c$, then $\{b,y,u_3\}$ is independent. Since $\{b,y,u_3\}$ dominates N[v], we arrive at a contradiction via Lemma 4.

Thus, c is not adjacent to u_3 and, by symmetry, b is not adjacent to u_4.

If $c \sim u_4$, then $\{b,c,u_3\}$ is independent and dominates N[v], a contradiction by Lemma 4. So c is not adjacent to u_4. By symmetry, b is not adjacent to u_3. Thus, $\{b,c,u_3\}$ is independent and dominates N[v]. We obtain a contradiction from Lemma 4.

Hence, the cyclic face configuration (3,m,3,n), m, n ≥ 4, cannot occur.

From Cases 1 through 4, we see that the only other possibility is that v has exactly one triangle in its face configuration.

Case 5. Suppose v has face configuration (3,l,m,n), l, m, n ≥ 4, with u_1u_2v as the face triangle at v. Since G is 3-connected, u_2 is not adjacent to u_3, u_3 is not adjacent to u_4 and u_4 is not adjacent to u_1.

Suppose $u_1 \sim u_3$. As in Case 4.1, we have $N(u_1) \supseteq N(u_4)$. Then by Lemma 5, it follows that $N(u_1) = N(u_4)$. But $u_1 \sim u_3$ and u_4

is not adjacent to u_3, a contradiction. Thus, u_1 is not adjacent to u_3. By symmetry, u_2 is not adjacent to u_4.

Let $w \sim u_3$, $w \notin \{v,u_1,u_2,u_4\}$. Suppose $w \sim u_4$. If w is not adjacent to u_1, then $\{w,u_1\}$ is independent and dominates N[v], a contradiction. So $w \sim u_1$ and, by symmetry, $w \sim u_2$. Thus, N(w) = N(v) by Lemma 5. Since $\delta \geq 4$ by Theorem 2, then $\{u_1,u_4\}$ is a cutset for G, contradicting 3-connectedness.

Hence, w is not adjacent to u_4 and so $N(u_3) \cap N(u_4) = \{v\}$.

Since G is planar and $\delta \geq 4$, then there exist points x and y such that $x \sim u_3$, $y \sim u_4$ and x is not adjacent to y, where $v \notin \{x,y\}$. Suppose $y \sim u_2$. If y is not adjacent to u_1, then $\{x,y,u_1\}$ is independent and dominates N[v], a contradiction. So $y \sim u_1$. But then $\{y,u_3\}$ is independent and dominates N[v], a contradiction. So y is not adjacent to u_2. By symmetry, x is not adjacent to u_1.

If y is not adjacent to u_1, then $\{x,y,u_1\}$ is independent and dominates N[v], a contradiction. So $y \sim u_1$ and, by symmetry, $x \sim u_2$.

Suppose $z \in N(u_2)-u_1$ implies $z \sim u_3$. Then $\{u_1,u_3\}$ dominates $N[u_2]$, $u_1 \sim u_2$ and u_3 is not adjacent to u_2. By Lemma 4, we obtain a contradiction. So there exists $z \in N(u_2)-u_1$ such that z is not adjacent to u_3.

Let a and b be neighbors of u_4 such that $\{a,b\} \cap \{v,y\} = \emptyset$, and let c and d be neighbors of u_3 such that $\{c,d\} \cap \{v,x\} = \emptyset$. From above, we know that $\{a,b,y\} \cap \{c,d,x\} = \emptyset$ (see Figure 3).

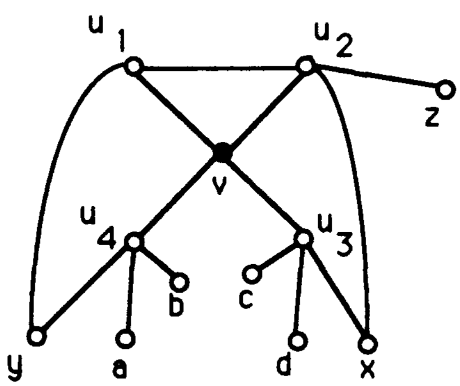

Figure 3

Suppose $a = z$ (that is, $a \sim u_2$). Also suppose $a \sim u_1$. Since $N(u_3) \cap N(u_4) = \{v\}$, then $\{a,u_3\}$ is independent. Also, $\{a,u_3\}$ dominates $N[v]$. We obtain a contradiction via Lemma 4.

So a is not adjacent to u_1. Suppose $a \sim t$ for all $t \in N(u_3) - v$. Then $\{a,v\}$ dominates $N[u_3]$, a contradiction. So there exists some $t \sim u_3$, $t \neq v$, such that t is not adjacent to a. Since G is planar, then $\{a,t,u_1\}$ is independent. Since also $\{a,t,u_1\}$ dominates $N[v]$, we obtain a contradiction via Lemma 4.

Thus, $\underline{a \neq z}$ and, by symmetry, $\underline{b \neq z}$.

Suppose there exists $s \in \{a,b\}$ such that $s \sim u_1$. Since G is planar, then either s is not adjacent to z or y is not adjacent to z. Say s is not adjacent to z. Then $\{s,z,u_3\}$ is independent and dominates $N[v]$, a contradiction. If y is not adjacent to z, then we obtain a similar contradiction.

Thus, <u>$s \in \{a,b\}$ implies s is not adjacent to u_1</u>. Likewise, <u>$t \in \{c,d\}$ implies t is not adjacent to u_2</u>.

If $y \sim c$ or $y \sim d$, then x is not adjacent to a. Then $\{a,x,u_1\}$ is independent and dominates $N[v]$, a contradiction. So <u>y is adjacent to neither c nor d</u>. But then $\{c,y,u_2\}$ is independent and dominates $N[v]$, a contradiction by Lemma 4.

Therefore, the face configuration $(3,l,m,n)$, $l, m, n \geq 4$, is not possible.

Hence if G has a point of degree four which is on a triangular face, then G is the octahedron graph given in Figure 1. []

Lemma 7. Suppose G is strongly well-covered planar and 3-connected. Then G cannot have a point of degree five with face configuration $(3,3,3,3,n)$, $n = 3, 4$, or 5.

<u>Proof</u>. Suppose G has a point v with $\deg(v) = 5$ and face configuration $(3,3,3,3,n)$, $n = 3$, 4 or 5. Let $N(v) = \{u_1,u_2,u_3,u_4,u_5\}$. Let $U_i = N(u_i)-N[v]$, for $i = 1, ..., 5$. Since u_i is in a triangle for all i, then it follows from Lemma 6 that <u>$\deg(u_i) \geq 5$</u> for all i.

Case 1. Suppose $n = 3$. Suppose $u_1 \sim u_3$. If $u_1 \sim u_4$, then $\{u_1\}$ dominates $N[v]$. By Lemma 4, we obtain a contradiction. So u_1 is not adjacent to u_4.

Suppose there exists $x \sim u_4$ such that x is not adjacent to u_1. Then $\{x,u_1\}$ is independent and dominates $N[v]$, $u_1 \sim v$ and x is not adjacent to v. By Lemma 4, we obtain a contradiction.

Thus, $N(u_1) \supseteq N(u_4)$. It follows from Lemma 5 that $N(u_1) = N(u_4)$. Since $u_1 \sim u_3$ and G is planar, then u_2 is not adjacent to u_4. But $u_1 \sim u_2$ implies $N(u_1) \neq N(u_4)$, a contradiction.

So u_1 is not adjacent to u_3. By symmetry, u_1 is not adjacent to u_4, u_2 is not adjacent to u_5, u_2 is not adjacent to u_4, and u_3 is not adjacent to u_5.

Case 1.1. Suppose $U_3 \cap U_4 \neq \emptyset$. Let $a \in U_3 \cap U_4$. If a is not adjacent to u_1, then $\{a,u_1\}$ is independent and dominates $N[v]$, a contradiction. So $a \sim u_1$.

Case 1.1.1. Suppose $a \sim u_2$. If a is not adjacent to u_5, then $\{a,u_5\}$ is independent and dominates $N[v]$, a contradiction; so $a \sim u_5$.

Suppose $x \in U_3$ implies $x \sim u_4$ (that is, $U_4 \supseteq U_3$). Then $\{u_1,u_4\}$ dominates $N[u_3]$, u_1 is not adjacent to u_3 and $u_4 \sim u_3$. By Lemma 4, we obtain a contradiction. Thus, there exists $x \in U_3$ such that x is not adjacent to u_4. Similarly, there exists $y \in U_4$ such that y is not adjacent to u_3.

If y is not adjacent to x, then $\{x,y,u_1\}$ is independent and dominates $N[v]$, a contradiction. So $y \sim x$ (see Figure 4). Since $\deg(u_2) \geq 5$, there exists $t \sim u_2$ such that $t \notin \{u_1,u_3,a,v\}$. In particular, $\{t,x,u_5\}$ is independent. Since $\{t,x,u_5\}$ also dominates $N[v]$, we obtain a contradiction by Lemma 4.

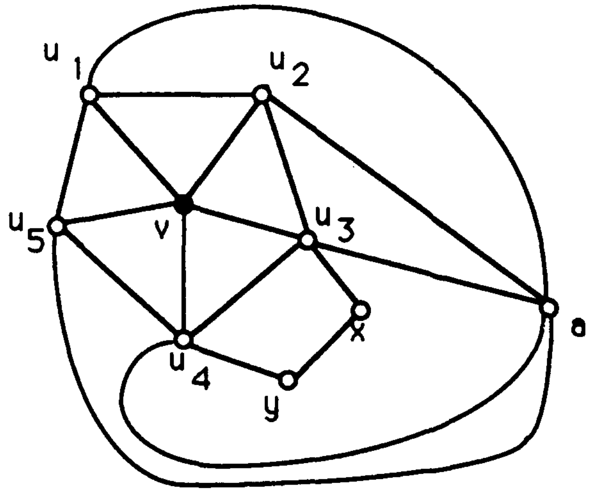

Figure 4

Case 1.1.2. Thus, a is not adjacent to u_2. By symmetry, a is not adjacent to u_5. Suppose $x \in U_2$ implies $x \sim a$. Then $\{a,v\}$ dominates

$N[u_2]$, $v \sim u_2$ and a is not adjacent to u_2. By Lemma 4, we obtain a contradiction.

Thus, there exists $x \in U_2$ such that x is not adjacent to a. But then $\{a,x,u_5\}$ is independent and dominates $N[v]$, a contradiction.

Case 1.2. Hence, $\underline{U_3 \cap U_4 = \emptyset}$. By symmetry, $\underline{U_i \cap U_{i+1} = \emptyset}$, for all i (addition mod 5). Since G is planar and $\deg(u_i) \geq 5$ for all i, then there exist $\underline{x \sim u_4}$ and $\underline{y \sim u_3}$ such that <u>x is not adjacent to y</u>. Suppose $x \sim u_1$. If $x \sim z$ for all $z \in U_5$, then $\{x,v\}$ is independent and dominates $N[u_5]$, $v \sim u_5$ and x is not adjacent to u_5. By Lemma 4, we obtain a contradiction. Thus, there exists $z \in U_5$ such that x is not adjacent to z. But then $\{x,z,u_3\}$ is independent and dominates $N[v]$, a contradiction.

So <u>x is not adjacent to u_1</u>. By symmetry, <u>y is not adjacent to u_1</u>. Thus, $\{x,y,u_1\}$ is independent and dominates $N[v]$, a contradiction.

So $n = 3$ is not possible.

Case 2. Suppose $n = 4$. Let the 4-face at v be vu_4au_5. If a is not adjacent to u_2, then $\{a,u_2\}$ is independent and dominates $N[v]$, a contradiction. So $\underline{a \sim u_2}$.

Suppose $a \sim u_1$. If a is not adjacent to u_3, then $\{a,u_3\}$ is independent and dominates $N[v]$, a contradiction. So $a \sim u_3$. Since $\deg(u_i) \geq 5$ for all i, there exist $x \sim u_4$ such that $x \notin \{a,v,u_3\}$ and $y \sim u_5$ such that $y \notin \{a,v,u_1\}$. Then $\{x,y,u_2\}$ is independent and dominates $N[v]$, a contradiction. Thus, <u>a is not adjacent to u_1</u>. By symmetry, <u>a is not adjacent to u_3</u>.

Suppose $x \in U_3$ implies $x \sim a$. Then $\{a,v\}$ dominates $N[u_3]$, $v \sim u_3$ and a is not adjacent to u_3. By Lemma 4, we have a contradiction. So there exists $\underline{x \in U_3}$ such that <u>x is not adjacent to a</u>. But then $\{a,x,u_1\}$ is independent (since G is planar) and dominates $N[v]$, a contradiction.

Hence, $n = 4$ is not possible.

Case 3. Suppose $n = 5$. Let the 5-face at v be vu_4abu_5. Since G is 3-connected, then <u>u_4 is not adjacent to u_5</u>, <u>b is not adjacent to u_4</u> and <u>a is not adjacent to u_5</u>.

Suppose u_4 and u_5 have a common neighbor w, $w \neq v$. If w is not adjacent to u_2, then $\{w,u_2\}$ is independent and dominates $N[v]$, a contradiction. So $w \sim u_2$. Since $\deg(u_3) \geq 5$, there exists $x \in U_3$ such that $x \neq w$. Since G is planar, $\{a,x,u_1\}$ is independent. Thus, $\{a,x,u_1\}$ is independent and dominates $N[v]$, a contradiction.

Hence, <u>u_4 and u_5 don't have a common neighbor w, $w \neq v$</u>.

Suppose $u_1 \sim u_3$. If u_1 is not adjacent to a, then $\{a,u_1\}$ is independent and dominates N[v], a contradiction. So $u_1 \sim a$. But then $\{b,u_3\}$ is independent and dominates N[v], a contradiction. Thus, u_1 is not adjacent to u_3.

Suppose $a \sim u_2$. Then $\{b,u_3\}$ is independent. If $b \sim u_1$, then $\{b,u_3\}$ dominates N[v], a contradiction. So b is not adjacent to u_1.

Suppose $x \in U_1$ implies $x \sim b$. Then $\{b,v\}$ dominates $N[u_1]$, $v \sim u_1$ and b is not adjacent to u_1. By Lemma 4, we obtain a contradiction. Thus, there exists $x \in U_1$ such that x is not adjacent to b. But then $\{b,x,u_3\}$ is independent and dominates N[v], a contradiction.

Hence, a is not adjacent to u_2; by symmetry, b is not adjacent to u_2.

Suppose $u_2 \sim u_4$. If b is not adjacent to u_2, then $\{b,u_2\}$ is independent and dominates N[v], a contradiction. So $b \sim u_2$. Let $z \sim u_3$ such that $z \notin \{u_2,u_4,v\}$. Since G is planar, then $\{a,z,u_1\}$ is independent. Since $\{a,z,u_1\}$ dominates N[v], we arrive at a contradiction via Lemma 4.

So u_2 is not adjacent to u_4; by symmetry, u_2 is not adjacent to u_5.

Suppose $x \in N(u_4)$-a implies $x \sim u_2$. Then $\{a,u_2\}$ dominates $N[u_4]$, $a \sim u_4$ and u_2 is not adjacent to u_4. By Lemma 4, we obtain a contradiction. So there exists $x \sim u_4$, $x \neq a$, such that x is not adjacent to u_2. Similarly, there exists $y \sim u_5$, $y \neq b$, such that y is not adjacent to u_2. From above, $x \neq y$. See Figure 5.

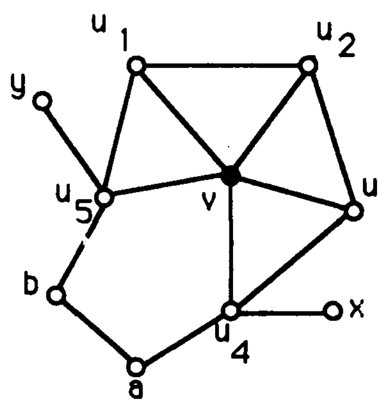

Figure 5

Suppose $x \sim y$. Since G is planar, then either x is not adjacent to b or y is not adjacent to a. Without loss of generality, assume x is not adjacent to b. Then $\{b,x,u_2\}$ is independent and dominates N[v], a contradiction from Lemma 4.

Thus, x is not adjacent to y. Then $\{x,y,u_2\}$ is independent and dominates N[v], a contradiction via Lemma 4.

Hence, n = 5 is not possible.

Thus, G cannot have a point v with deg(v) = 5 and face configuration (3,3,3,3,n), n = 3, 4 or 5. []

Lebesgue [8] developed the theory of Euler contributions for planar graphs. The Euler contribution of a vertex v, $\phi(v)$, is defined as the quantity $\phi(v) = 1 - (1/2)\deg(v) + \Sigma(1/x_i)$, where the sum is taken over all faces F_i incident to v and x_i is the size of F_i. If $|F(G)|$ denotes the number of faces in the plane graph G, then it follows that $\Sigma_v \phi(v) = |V(G)| - |E(G)| + |F(G)|$. Here the sum is taken over all vertices v in G. Since Euler's formula for plane graphs says $|V(G)| - |E(G)| + |F(G)| = 2$, then we have $\Sigma_v \phi(v) = 2$. Thus, $\phi(v)$ must be positive for some v in G. From the definition of $\phi(v)$, it follows easily that $\phi(v) \leq 0$ whenever $\deg(v) \geq 6$. Thus, if $\phi(v) > 0$, then $\deg(v) \leq 5$.

As a consequence of the two previous lemmas and the theory of Euler contributions, we find all 3-connected planar strongly well-covered graphs in the following theorem.

Theorem 8. Suppose G is strongly well-covered planar and 3-connected. Then G is the octahedron graph shown in Figure 1.

Proof. From Theorem 2, $\delta \geq 4$. Suppose v is a point in G with *positive* Euler contribution; that is, $\phi(v) > 0$. Then deg(v) = 4 or 5.

If deg(v) = 4, then $\phi(v) = 1 - (1/2)(4) + \Sigma(1/x_i) = -1 + \Sigma(1/x_i)$, where the sum is taken over all faces incident to v. For $\phi(v)$ to be positive, $\Sigma(1/x_i)$ must be greater than 1. Thus, v must lie on a triangular face in order for $\phi(v)$ to be positive. From Lemma 6, this can only occur if G is the graph given in Figure 1.

If deg(v) = 5, then $\phi(v) = 1 - (1/2)(5) + \Sigma(1/x_i) = -3/2 + \Sigma(1/x_i)$, where the sum is taken over all faces incident to v. For $\phi(v)$ to be positive in this case, $\Sigma(1/x_i)$ must be greater than 3/2. Thus, v must have a face configuration of the form (3,3,3,3,n), n = 3, 4 or 5. But from Lemma 7, this cannot occur. []

From Theorem 3, we know that all strongly well-covered graphs on more than four points are 3-connected. Thus, we conclude in the following corollary that there are exactly four planar strongly well-covered graphs.

Corollary 9. The only planar strongly well-covered graphs are K_1, K_2, C_4 and the octahedron graph shown in Figure 1.

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